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ON INTEGRATION OF KINEMATIC EQUATIONS OF A RIGID BODY'S SCREW-MOTION *

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The kinematic equations of a rigid body screw-motion are examined in parameters the complex combinations of which are components of the biquaternion of the body screw-motion. The structure of the general solution of the kinematic equations is established, and a case of their integrability in quadratures is investigated.

1. As is well known, an arbitrary spatial motion of a rigid body is equivalent to a screw-motion. We introduce two coordinate systems: system $O_2Y_1Y_2Y_3(Y)$ attached to the body and a support system $O_1X_1X_2X_3(X)$, the two coinciding at the initial position. A finite displacement of the attached coordinate system Y relative to the support system X is defined by the dual vector of finite screw-motion /1/

$$\boldsymbol{\Theta} = 2\mathbf{E}\,\mathrm{tg}\,(\Phi/2)$$

Here $\mathbf{E}$ is a unit screw turn of axis ab of the screw-motion, $\Phi = \varphi + s\varphi^\circ$ is the dual angle of turning of the body, φ is the ordinary angle of turn of the body about the axis ab, φ° is the translational motion of the body along this axis, s is the Clifford symbol, $s^2 = 0$. We associate with the screw-motion $\boldsymbol{\Theta}$ the eigenbiquaternion $\boldsymbol{\Lambda}$, i.e., a biquaternion /1,2/ whose components are the dual Rodrigues—Hamilton parameters $\Lambda_j\ (j = 0, 1, 2, 3)$

$$\boldsymbol{\Lambda} = \Lambda_0 \mathbf{1} + \Lambda_1 \mathbf{i}_1 + \Lambda_2 \mathbf{i}_2 + \Lambda_3 \mathbf{i}_3$$

Here $\mathbf{1}, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are unit vectors of a hypercomplex space /3/, and the quantities Λ_j are dual analogs of the real Rodrigues—Hamilton parameters /3—5/ defined by the relations

$$\Lambda_0 = \cos(\Phi/2), \quad \Lambda_i = \sin(\Phi/2)\cos\Gamma_i \quad (i = 1, 2, 3)$$

in which $\Gamma_i = \gamma_i + s\gamma_i^\circ$ is the dual angle between the axis of screw-motion $\boldsymbol{\Theta}$ and the axis O_1X_i (O_2Y_i) /1/, γ_i is the ordinary angle between axes ab and O_1X_i, and γ_i° is the shortest distance between axes ab and O_1X_i.

Using the expressions for the trigonometic functions of a dual angle /1/, we represent the parameters $\Lambda_j\ (j = 0, 1, 2, 3)$ as complex combinations of the real quantities λ_j and $\lambda_j^\circ\ (j = 0, 1, 2, 3)$

$$\Lambda_j = \lambda_j + s\lambda_j^\circ \ (j = 0, 1, 2, 3) \tag{1.1}$$

Here λ_j are real Rodrigues—Hamilton parameters defined by the relations /3—5/

$$\lambda_0 = \cos(\varphi/2), \quad \lambda_i = \sin(\varphi/2)\cos\gamma_i \ (i = 1, 2, 3)$$

The quantities λ_j° are defined by the relations

$$\lambda_0^\circ = \varphi^\circ \frac{\partial \lambda_0}{\partial \varphi} = -\frac{1}{2}\varphi^\circ \sin\frac{\varphi}{2} \tag{1.2}$$

$$\lambda_i^\circ = \varphi^\circ \frac{\partial \lambda_i}{\partial \varphi} + \gamma_i^\circ \frac{\partial \lambda_i}{\partial \gamma_i} = \frac{1}{2}\varphi^\circ \cos\frac{\varphi}{2}\cos\gamma_i - \gamma_i^\circ \sin\frac{\varphi}{2}\sin\gamma_i$$

$$(i = 1, 2, 3)$$

We call the quantities λ_j and $\lambda_j^\circ\ (j = 0, 1, 2, 3)$ the parameters of the rigid body screw-motion. With due regard to equalities (1.1) the eigenbiquaternion $\boldsymbol{\Lambda}$ assumes the form

$$\boldsymbol{\Lambda} = \boldsymbol{\lambda} + s\boldsymbol{\lambda}^\circ, \quad \boldsymbol{\lambda} = \lambda_0 + \boldsymbol{\lambda}_v, \quad \boldsymbol{\lambda}_v = \lambda_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3$$

$$\boldsymbol{\lambda}^\circ = \lambda_0^\circ + \boldsymbol{\lambda}_v^\circ, \quad \boldsymbol{\lambda}_v^\circ = \lambda_1^\circ\mathbf{i}_1 + \lambda_2^\circ\mathbf{i}_2 + \lambda_3^\circ\mathbf{i}_3$$

Here $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\circ$ are eigenquaternions. The generator of the instantaneous screw-motion velocity $\mathbf{U}$ (the kinematic screw) of the ridid body, relative to pole O_2 (we can take for it, for example, the body center of mass) is equal to the dual vector $\boldsymbol{\omega} + s\mathbf{v}$ /1/. Here $\mathbf{v}$ is the velocity vector of point O_2 of the body relative to basis X, $\boldsymbol{\omega}$ is the body angular rotation velocity vector in basis X. Therefore, the dual orthogonal projections $U_i\ (i = 1, 2, 3)$ of the kinematic screw $\mathbf{U}$ on the axis of the attached coordinate system are

$$U_i = \omega_i + sv_i \quad (i = 1, 2, 3) \tag{1.3}$$

where ω_i and v_i are the projections of vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ on the attached axis O_2Y_i.

To obtain the kinematic equations of the rigid body screw-motion, which establish the dependence between the dual Rodrigues—Hamilton parameters, their derivatives, and the dual

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orthogonal projections of the kinematic screw on the axes of the attached trihedron, we apply the Kotel'nikov—Study transference principle /1/ to the kinematic equations of the body spherical motion in terms of real Rodrigues—Hamilton parameters /3—5/. We write the equations thus obtained in two equivalent matrix forms

$$2\Theta^{\cdot} = N_u\Theta, \quad 2N^{\cdot} = N_u N \tag{1.4}$$

$$\Theta^T = \| \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3 \|, \quad \Theta^{\cdot T} = \| \Lambda_0^{\cdot}, \Lambda_1^{\cdot}, \Lambda_2^{\cdot}, \Lambda_3^{\cdot} \|$$

$$N = \begin{Vmatrix} \Lambda_0 & -\Lambda_1 & -\Lambda_2 & -\Lambda_3 \\ \Lambda_1 & \Lambda_0 & \Lambda_3 & -\Lambda_2 \\ \Lambda_2 & -\Lambda_3 & \Lambda_0 & \Lambda_1 \\ \Lambda_3 & \Lambda_2 & -\Lambda_1 & \Lambda_0 \end{Vmatrix}, \quad N_u = \begin{Vmatrix} 0 & -U_1 & -U_2 & -U_3 \\ U_1 & 0 & U_3 & -U_2 \\ U_2 & -U_3 & 0 & U_1 \\ U_3 & U_2 & -U_1 & 0 \end{Vmatrix}$$

Here the dot denotes differentiation with respect to time t and the index T is the transposition symbol. The kinematic Eqs. (1.4) of the body screw-motion are dual matrix homogeneous linear differential equations with variable coefficients. In the first of Eqs. (1.4) we pass from the dual Rodrigues—Hamilton parameters to the real parameters of the body screw-motion using Eqs. (1.1) and (1.3). This results in splitting the dual equation into two real equations

$$2\theta^{\cdot} = n_\omega\theta, \quad 2\theta^{\circ\cdot} = n_\omega\theta^\circ + n_v\theta \tag{1.5}$$

$$\theta^T = \| \lambda_0, \lambda_1, \lambda_2, \lambda_3 \|, \quad \theta^{\circ T} = \| \lambda_0^\circ, \lambda_1^\circ, \lambda_2^\circ, \lambda_3^\circ \|$$

Here the matrices n_ω and n_v have the structure of matrix N_u and are composed of the projections ω_i and v_i $(i = 1, 2, 3)$ of vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ on the attached basis.

The first of Eqs. (1.5) is the matrix kinematic equation of the body spherical motion about point O_2 in real Rodrigues—Hamilton parameters. It is independent of the second equation. The second of Eqs. (1.5) depends on the first but, in constrast to it, is inhomogeneous, defining the translational motion of the body together with pole O_2. Equations (1.5), as well as each of the dual equations (1.4), enable us to determine the body's screw-motion from specified projections of vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ on the attached basis and from specified initial conditions for parameters λ_j and λ_j°. The Rodrigues—Hamilton parameters λ_j define the body orientation in the support coordinate system, to determine the translational motion of the body it is necessary to use the following formulas

$$x_i, y_i = 2\left(\lambda_0\lambda_i^\circ - \lambda_0^\circ\lambda_i \pm \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\lambda_j\lambda_k^\circ \right) \quad (i = 1, 2, 3) \tag{1.6}$$

Here x_i and y_i $(i = 1, 2, 3)$ are the projections of the radius vector $\mathbf{r}$ drawn from the origin O_1 of the coordinate system X to pole O_2 on the axis of the support basis X and of the attached basis Y; ε_{ijk} is the Levi-Civita symbol /4/; the plus sign corresponds to x_i and the minus sign to y_i. In quaternion notation formulas (1.6) become

$$\mathbf{r}_X = 2\boldsymbol{\lambda}^\circ \circ \boldsymbol{\lambda}^* = 2(\lambda_0\boldsymbol{\lambda}_v^\circ - \lambda_0^\circ\boldsymbol{\lambda}_v + \boldsymbol{\lambda}_v \times \boldsymbol{\lambda}_v^\circ) \tag{1.7}$$

$$\mathbf{r}_Y = 2\boldsymbol{\lambda}^* \circ \boldsymbol{\lambda}^\circ = 2(\lambda_0\boldsymbol{\lambda}_v^\circ - \lambda_0^\circ\boldsymbol{\lambda}_v - \boldsymbol{\lambda}_v \times \boldsymbol{\lambda}_v^\circ)$$

Here $\mathbf{r}_X = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$ and $\mathbf{r}_Y = y_1\mathbf{i}_1 + y_2\mathbf{i}_2 + y_3\mathbf{i}_3$ represent hypercomplex mapping of vector $\mathbf{r}$ onto the support and the attached bases /3/; The quaternion $\boldsymbol{\lambda}^*$ is adjoint to quaternion $\boldsymbol{\lambda}$: $\boldsymbol{\lambda}^* = \lambda_0 - \boldsymbol{\lambda}_v$; the symbol $\circ$ denotes quaternion multiplication; $\boldsymbol{\lambda}_v$ and $\boldsymbol{\lambda}_v^\circ$ are the vector-valued parts of quaternions $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\circ$; the scalar-valued parts of quaternions $\boldsymbol{\lambda}^\circ \circ \boldsymbol{\lambda}^*$ and $\boldsymbol{\lambda}^* \circ \boldsymbol{\lambda}^\circ$ are zero, since $\lambda_0\lambda_0^\circ + \lambda_1\lambda_1^\circ + \lambda_2\lambda_2^\circ + \lambda_3\lambda_3^\circ = 0$.

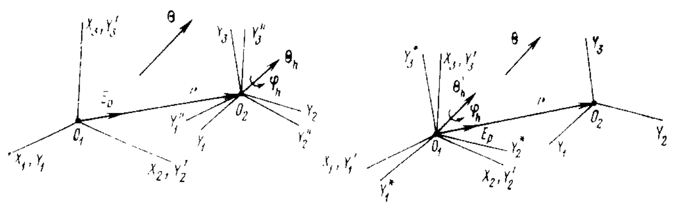

Fig.1

2. Let us prove formulas (1.7). A finite displacement of the body is equivalent to one of two sequences of motions (Fig.1):

1) the sequence of the body translational motion at the velocity of pole O_2, defined by the screw-motion $\boldsymbol{\Theta}_p = 2\mathbf{E}_p \operatorname{tg}(\Phi_p/2) = 2\mathbf{E}_p \operatorname{tg}(s\varphi_p^\circ/2)$, and of the body spherical motion about pole O_2, defined by the screw-motion $\boldsymbol{\Theta}_h = 2\mathbf{E}_h \operatorname{tg}(\Phi_h/2) = 2\mathbf{E}_h \operatorname{tg}(\varphi_h/2)$;

2) the sequence of the body spherical motion about pole O_2, defined by the screw-motion $\boldsymbol{\Theta}_h' = 2\mathbf{E}_h' \operatorname{tg}(\varphi_h/2)$, and of the body translational motion defined by the screw-motion $\boldsymbol{\Theta}_p$.

We illustrate what we have said by the conventional scheme

$$X \xrightarrow{\boldsymbol{\Theta}} Y \backsim X \xrightarrow{\boldsymbol{\Theta}_p} Y'' \xrightarrow{\boldsymbol{\Theta}_h} Y \backsim X \xrightarrow{\boldsymbol{\Theta}_h'} Y^* \xrightarrow{\boldsymbol{\Theta}_p} Y$$

Here $\backsim$ is the equivalence symbol. With the screw-motions $\boldsymbol{\Theta}_p$ and $\boldsymbol{\Theta}_h$ for the first sequence of displacements and the screw-motions $\boldsymbol{\Theta}_h'$ and $\boldsymbol{\Theta}_p$ for the second we associate the eigenbiquaternions $\mathbf{P}$, $\mathbf{H}$ and $\mathbf{H}'$, $\mathbf{P}'$

$$\mathbf{P} = 1 + s\mathbf{p}^\circ, \quad \mathbf{P}' = 1 + s\mathbf{p}^{\circ\prime}, \quad \mathbf{H} = \mathbf{h}, \quad \mathbf{H}' = \mathbf{h}'$$
$$\mathbf{z} = z_0 + z_1\mathbf{i}_1 + z_2\mathbf{i}_2 + z_3\mathbf{i}_3; \quad \mathbf{z} = \mathbf{p}^\circ, \mathbf{p}^{\circ\prime}, \mathbf{h}, \mathbf{h}'$$

By the Kotel'nikov—Study transference principle /1/ the eigenbiquaternion Λ of the resulting finite displacement is determined in terms of the eigenbiquaternions of the component displacements by a rule that is the dual analog of the rule for finding the eigenquaternion of the resulting turn from the eigenquaternions of the component turns /3/. Therefore $\Lambda = \mathbf{P}\circ\mathbf{H} = \mathbf{H}'\circ\mathbf{P}'$ and consequently,

$$\boldsymbol{\lambda} + s\boldsymbol{\lambda}^\circ = (1 + s\mathbf{p}^\circ) \circ \mathbf{h} = \mathbf{h}' \circ (1 + s\mathbf{p}^{\circ\prime})$$

Hence we obtain

$$\mathbf{h} = \mathbf{h}' = \boldsymbol{\lambda}, \quad \mathbf{p}^\circ = \boldsymbol{\lambda}^\circ \circ \boldsymbol{\lambda}^*, \quad \mathbf{p}^{\circ\prime} = \boldsymbol{\lambda}^* \circ \boldsymbol{\lambda}^\circ \tag{2.1}$$

The components p_j° and $p_j^{\circ\prime}$ $(j = 0, 1, 2, 3)$ of quaternions $\mathbf{p}^\circ$ and $\mathbf{p}^{\circ\prime}$ are found from formulas analogous to (1.2). It should be born in mind that for the motion $\boldsymbol{\Theta}_p$ being examined the translational motion $\varphi_p^\circ = r$, the angle of turn φ_p, and the shortest distance β_i° $(\beta_i^{\circ\prime})$ between axis O_1X_i (O_2Y_i) and the axis of screw-motion $\boldsymbol{\Theta}_p$ are zero, and the direction cosines $\cos\beta_i$ and $\cos\beta_i'$ $(i = 1, 2, 3)$ between the axis of screw-motion $\boldsymbol{\Theta}_p$ and the axes of the support basis X and of the connected basis Y equal the corresponding direction cosines of vector $\mathbf{r}$ in the same bases because the axis of screw-motion $\boldsymbol{\Theta}_p$ passes through the origin of coordinate systems X and Y and coincides with the straight line on which vector $\mathbf{r}$ lies. Therefore

$$\mathbf{p}^\circ = \frac{1}{2}\sum_{k=1}^{3} x_k\mathbf{i}_k, \quad \mathbf{p}^{\circ\prime} = \frac{1}{2}\sum_{k=1}^{3} y_k\mathbf{i}_k \tag{2.2}$$
$$x_k = r\cos\beta_k, \quad y_k = r\cos\beta_k'$$

From (2.2) we see that the hypercomplex images $\mathbf{r}_X$ and $\mathbf{r}_Y$ of vector $\mathbf{r}$ are related to quaternions $\mathbf{p}^\circ$ and $\mathbf{p}^{\circ\prime}$ by the equalities $\mathbf{r}_X = 2\mathbf{p}^\circ$, $\mathbf{r}_Y = 2\mathbf{p}^{\circ\prime}$. Then allowing for (2.1), we obtain formulas (1.7). Note that, if one of the hyper-complex images of vector $\mathbf{r}$, say $\mathbf{r}_X$, has been established, its other hypercomplex image $\mathbf{r}_Y$ can be found by using the following coordinate transformation rule for a fixed vector on the support and attached bases /3/

$$\mathbf{r}_Y = \boldsymbol{\lambda}^* \circ \mathbf{r}_X \circ \boldsymbol{\lambda}$$

Having substituted the equality $\mathbf{r}_X = 2\boldsymbol{\lambda}^\circ\circ\boldsymbol{\lambda}^*$ into this relation, we arrive at an expression for $\mathbf{r}_Y$, which coincides with the second expression obtained in (1.7).

3. Let us consider the integration of the first matrix dual kinematic equation in (1.4) for the body screw-motion. In this equation we assume that the elements $U_i = \omega_i + sv_i$ of the dual matrix of coefficients N_u are known time functions, since the projections ω_i and v_i of vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ on the axis attached to the body can either be measured or be obtained, for example, from a navigation system. The structure of the general solution for the first equation in (1.5) has been indicated in /6/. On the basis of the Kotel'nikov—Study transference principle we extend this result to the first equation in (1.4), which is the dual analog of the first equation in (1.5). As a result we obtain the following structure for the general solution $\Theta = \Theta(t)$ of the first of Eqs. (1.4).

$$\Theta = N^+\Theta_0 \tag{3.1}$$

Here $\Theta_0^T = \Theta^T(t_0) = \|\Lambda_{00}, \Lambda_{10}, \Lambda_{20}, \Lambda_{30}\|^T$ and N^+ is the matrizant of the second dual matrix equation in (1.4). It is made up of elements Λ_j^+ $(j = 0, 1, 2, 3)$ and has the structure of matrix N.

Applying the Kotel'nikov—Study transference principle to another result in /6/, concerning the integrability of the first equation in (1.5), we establish that the first equation in (1.4) can be integrated in quadratures when the kinematic screw

$$\mathbf{U} = F(t)\,[D_1\mathbf{E}_1' + Q_1(\cos(A + B)\,\mathbf{E}_2' + \sin(A + B)\,\mathbf{E}_3')] \tag{3.2}$$
$$A(t) = A_1\int_{t_0}^{t} F(t)\,dt$$

Here $\mathbf{E}_i'$ $(i = 1, 2, 3)$ are unit screw-turns of the attached coordinate system; $D_1 = d_1 + sd_1^\circ$, $Q_1 = q_1 + sq_1^\circ$, $A_1 = \alpha_1 + s\alpha_1^\circ$, $B = \beta + s\beta^\circ$ are certain dual constants; $F(t) = f(t) + sf^\circ(t)$ is an arbitrary dual time function bounded and integrable on the interval $[t_1, t_2]$ being examined. To a kinematic screw of form (3.2) correspond the body angular rotation velocity vector $\boldsymbol{\omega}$ and the velocity vector $\mathbf{v}$ of pole O_2, of the form ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors of the connected trihedron

$$\boldsymbol{\omega} = f(t)[d_1\mathbf{e}_1' + q_1(\mathbf{e}_2'\cos\tau + \mathbf{e}_3'\sin\tau)]$$
$$\mathbf{v} = a'\mathbf{e}_1' + (b'\cos\tau - c'\sin\tau)\mathbf{e}_2' + (b'\sin\tau + c'\cos\tau)\mathbf{e}_3'$$
$$a'(t) = d_1^\circ f(t) + d_1 f^\circ(t), \quad b'(t) = q_1^\circ f(t) + q_1 f^\circ(t)$$
$$\tau(t) = \alpha_1\int_{t_0}^{t} f(t)\,dt + \beta, \quad c'(t) =$$

$$q_1 f(t)\left[\alpha_1^\circ \int_{t_0}^{t} f(t)\,dt + \alpha_1 \int_{t_0}^{t} f^\circ(t)\,dt + \beta^\circ\right]$$

The general solution of the first of Eqs. (1.4) for a kinematic screw of form (3.2) is given by relation (3.1) in which the elements $\Lambda_j^+ \, (j = 0, 1, 2, 3)$ of matrix N^+ are

$$\begin{aligned}
&\Lambda_0^+ = \cos\frac{A}{2}\cos\frac{K}{2} + \frac{D_1 + A_1}{U'}\sin\frac{A}{2}\sin\frac{K}{2} \\
&\Lambda_1^+ = \frac{D_1 + A_1}{U'}\cos\frac{A}{2}\sin\frac{K}{2} - \sin\frac{A}{2}\cos\frac{K}{2} \\
&\Lambda_2^+ = \frac{Q_1}{U'}\cos\left(\frac{A}{2} + B\right)\sin\frac{K}{2}, \quad \Lambda_3^+ = \frac{Q_1}{U'}\sin\left(\frac{A}{2} + B\right)\sin\frac{K}{2} \\
&K = U'\int_{t_0}^{t} F(t)\,dt, \quad U' = [(D_1 + A_1)^2 + Q_1^2]^{1/2}
\end{aligned} \tag{3.3}$$

This general solution is the dual analog of the general solution obtained in /6/ for the first of Eqs. (1.5).

The following are particular cases of the integrability of the first equation in (1.4).

1) The kinematic screw $\mathbf{U}$ of the body performs a conic motion relative to the attached coordinate system. This case obtains if we set $F(t) = 1$ and $B = 0$ in the expression (3.2) for the kinematic screw. Then

$$\begin{aligned}
&\mathbf{U} = D_1\mathbf{E}_1' + Q_1\{\cos[A_1(t - t_0)]\mathbf{E}_2' + \sin[A_1(t - t_0)]\mathbf{E}_3'\} \\
&\boldsymbol{\omega} = d_1\mathbf{e}_1' + q_1\{\cos[\alpha_1(t - t_0)]\mathbf{e}_2' + \sin[\alpha_1(t - t_0)]\mathbf{e}_3'\} \\
&\mathbf{v} = d_1^\circ\mathbf{e}_1' + \{q_1^\circ\cos[\alpha_1(t - t_0)] - q_1\alpha_1^\circ(t - t_0)\sin[\alpha_1(t - t_0)]\}\mathbf{e}_2' + \\
&\quad \{q_1^\circ\sin[\alpha_1(t - t_0)] + q_1\alpha_1^\circ(t - t_0)\cos[\alpha_1(t - t_0)]\}\mathbf{e}_3'
\end{aligned}$$

To obtain the general solution of the first equation in (1.4) we also have to set $F(t) = 1$ and $B = 0$ in (3.3) as well.

2) The kinematic screw $\mathbf{U}$ of the body retains a fixed position in the attached coordinate system, changing only in modulus. In this case the kinematic screw $\mathbf{U}$ is obtained from (3.2) with $A_1 = 0$

$$\mathbf{U} = F(t)\{D_1\mathbf{E}_1' + Q_1[\mathbf{E}_2'\cos B + \mathbf{E}_3'\sin B]\} \tag{3.4}$$

To this kinematic screw correspond the body angular velocity vector $\boldsymbol{\omega}$ and the velocity vector $\mathbf{v}$ of pole O_2 of the body, which are of the form

$$\begin{aligned}
&\boldsymbol{\omega} = f(t)[d_1\mathbf{e}_1' + q_1(\mathbf{e}_2'\cos\beta + \mathbf{e}_3'\sin\beta)] \\
&\mathbf{v} = [d_1 f(t) + d_1 f^\circ(t)]\mathbf{e}_1' + (a'\cos\beta - b'\sin\beta)\mathbf{e}_2' + (a'\sin\beta + b'\cos\beta)\mathbf{e}_3' \\
&a'(t) = q_1^\circ f(t) + q_1 f^\circ(t), \quad b'(t) = q_1\beta^\circ f(t)
\end{aligned} \tag{3.5}$$

Let us consider in detail the conditions to be imposed on vectors $\boldsymbol{\omega}$ and $\mathbf{v}$, under which the kinematic screw of the body retains its position in the attached basis but can change its modulus. The dual direction cosines of the screw $\mathbf{U}$ of form (3.4) in the axes of the attached base are defined by the equalities

$$\cos B_i = U_i/U = C_i = c_i + sc_i^\circ \quad (i = 1, 2, 3) \tag{3.6}$$

Here $U = (U_1^2 + U_2^2 + U_3^2)^{1/2}$ is the dual modulus of screw $\mathbf{U}$, C_i are dual constants expressed in terms of the dual constants D_1, Q_1, B; c_i and c_i° are real constants defined in terms of constants $d_1, q_1, \beta, d_1^\circ, q_1^\circ, \beta^\circ$. From (3.6) and (1.3) we obtain

$$\cos B_i = c_i + sc_i^\circ = \omega^{-1}\omega_i + s[\omega^{-1}v_i - \omega^{-3}(\boldsymbol{\omega}\cdot\mathbf{v})\omega_i], \quad \omega = |\boldsymbol{\omega}| \neq 0 \tag{3.7}$$

From (3.7) we find

$$\omega^{-1}\boldsymbol{\omega} = \mathbf{c}, \quad \omega^{-1}\mathbf{v} - \omega^{-3}(\boldsymbol{\omega}\cdot\mathbf{v})\boldsymbol{\omega} = \mathbf{c}^\circ \tag{3.8}$$

Here $\mathbf{c}$ and $\mathbf{c}^\circ$ are vectors fixed in the attached coordinate system, whose projections on the axis of this basis are equal c_i and $c_i^\circ \, (i = 1, 2, 3)$. Scalar multiplication of both sides of the second of Eqs. (3.8) by vector $\boldsymbol{\omega}$, yields $\mathbf{c}^\circ\cdot\boldsymbol{\omega} = 0$. Since generally , $|\mathbf{c}^\circ| \neq 0$ and $|\boldsymbol{\omega}| \neq 0$ hence vectors $\mathbf{c}^\circ$ and $\boldsymbol{\omega}$ are perpendicular. We construct the vector

$$\omega^{-1}\mathbf{v} = \mathbf{c}^\circ + \boldsymbol{\eta}, \quad \boldsymbol{\eta} = \omega^{-3}(\boldsymbol{\omega}\cdot\mathbf{v})\boldsymbol{\omega}$$

Vector $\mathbf{c}^\circ$ is constant in magnitude and retains its direction in the attached basis. In this basis the vector $\boldsymbol{\eta}$ is constant in direction but variable in modulus. The hodograph of vector $\omega^{-1}\mathbf{v}$ is a straight line parallel to vector $\boldsymbol{\omega}$ lying in the plane formed by vectors $\mathbf{c}^\circ$ and $\boldsymbol{\eta}(\boldsymbol{\omega})$. This plane is fixed relative to the attached coordinate system. In the general case the vector $\mathbf{v}$, whose in direction coincides with that of vector $\omega^{-1}\mathbf{v}$, generally changes its orintation relative to the attached system. Thus, in order that the kinematic screw $\mathbf{U}$ of the body retains its position in the attached system, while being able to change in modulus, it is necessary and sufficient that the body angular velocity vector $\boldsymbol{\omega}$ retains its direction in the attached coordinate system, i.e., that the first equality in (3.8) be fulfilled and that the velocity vector $\mathbf{v}$ of the body pole O_2 satisfies the second equation in (3.8), in which $\mathbf{c}^\circ$ is some vector constant in the attached coordinate system, and is perpendicular to vector $\boldsymbol{\omega}$. In this case the hodograph of vector $\omega^{-1}\mathbf{v}$ is a straight line parallel to vector $\boldsymbol{\omega}$ and vector $\mathbf{v}$, in contrast to vector $\boldsymbol{\omega}$, can change not only its modulus but also its direction with respect to the attached trihedron.

Note that for the fulfilment of conditions (3.8) it is sufficient that vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ retain their orientation unchanged in the attached basis and their moduli be directly proportional. The expressions for vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ in this particular case are obtained from formulas (3.5) with $f(t) = f^\circ(t)$. The general solution of the first of Eqs. (1.4) for a kinematic screw of form (3.4) is given by relation (3.1). The dual elements Λ_j^+ of the matrix N^+ in (3.1) are obtained from formulas (3.3) with $A_1 = 0$.

Passing from the dual quantities to real ones, we find that the real Rodrigues—Hamilton parameters define the body's orientation in the support basis are determined in this case by the matrix relation

$$\| \lambda_0, \lambda_1, \lambda_2, \lambda_3 \|^T = \| \lambda_{00}, \lambda_{10}, \lambda_{20}, \lambda_{30} \| \, n^{+T}$$

in which the matrix n^+ is composed of elements

$$\lambda_0^+ = \cos \frac{1}{2} \int_{t_0}^{t} \omega \, dt, \quad \lambda_i^+ = \omega^{-1}\omega_i \sin \frac{1}{2} \int_{t_0}^{t} \omega \, dt \quad (i = 1, 2, 3) \tag{3.9}$$

The moment-valued parts $\lambda_j^{\circ+}$ of the dual parameters Λ_j^+, that define the body translational motion, have the form

$$\lambda_0^{\circ+} = -{}^1/_2\, \varphi^\circ \sin(\varphi/2) \tag{3.10}$$

$$\lambda_i^{0+} = {}^1/_2\, \varphi^\circ \omega^{-1}\omega_i \cos(\varphi/2) + [\omega^{-1}v_i - \omega^{-3}(\boldsymbol{\omega}\cdot\mathbf{v})\,\omega_i] \sin(\varphi/2)$$

$$\varphi = \int_{t_0}^{t} \omega \, dt, \quad \varphi^\circ = \int_{t_0}^{t} \omega^{-1}(\boldsymbol{\omega}\cdot\mathbf{v}) \, dt$$

Using the second formula in (1.7) and the rule for finding the eigenbiquaternion of the resulting displacement in terms of the eigenbiquaternions of the component displacements, we find that

$$\mathbf{r}_Y = \mathbf{r}_{0Y} + 2\boldsymbol{\lambda}^{+*}\circ\boldsymbol{\lambda}^{\circ+} \tag{3.11}$$

Here $\mathbf{r}_{0Y}$ is the hypercomplex mapping of vector $\mathbf{r}_0 = \mathbf{r}(t_0)$ onto the connected basis; $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^{\circ+}$ are quaternions whose components are the quantities λ_j^+ and $\lambda_j^{\circ+}$ $(j = 0, 1, 2, 3)$. As follows from (3.9) and (3.10), the vector-valued parts of quaternions $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^{\circ+}$ are defined in the attached basis. The hypercomplex images $\mathbf{r}_Y$ and $\mathbf{r}_{0Y}$ are defined in this same basis. Therefore, the unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ of the hypercomplex space can be combined with the unit vectors of the attached basis. Equality (3.11) then becomes a usual vector equality. Transforming it with due regard to (3.8)—(3.10), we obtain

$$\mathbf{r} = \mathbf{r}_0 + \varphi^\circ \mathbf{c} + \sin\varphi\, \mathbf{c}^\circ + 2\sin^2(\varphi/2)\, \mathbf{c}^\circ \times \mathbf{c} \tag{3.12}$$

Thus, Eqs. (3.9) and (3.12) describe the motion of the rigid body when its kinematic screw $\mathbf{U}$ is of form (3.4), i.e., when vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ satisfy conditions (3.8).

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